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Lower bounds for critical values of a cancellative model

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Abstract. We consider the following one-dimensional discrete-time cancellative model whose evolution is given by $\eta_{n+1}(x) = \eta_n(x+1) + \eta_n(x-1)$ (mod 2) with probability p and $\eta_{n+1} = 0$ with probability $1-p$. Concerning critical probabilities p_c and p_c^* on a survival probability, it is known that $0.706 \leqslant p_c \leqslant p_c^* < 1$ under a condition. In this paper, we give improved lower bounds of 0.771 and 0.781 on p_c and p_c^*, respectively, by finding suitable supermartingales for the model.

1. Introduction

Here, we consider the following one-dimensional discrete-time process η_n^A at time n starting from $A \subset 2\mathbf{Z}$ whose evolution satisfies:

(a) $P(x \in \eta_{n+1}^A | \eta_n^A) = f(|\eta_n^A \cap \{x-1, x+1\}|)$,
(b) given η_n^A, the events $\{x \in \eta_{n+1}^A\}$ are independent, where

$$f(0) = 0 \qquad f(1) = p \qquad \text{and} \qquad f(2) = 0.$$

This process can be considered on a space $S = \{s = (n, x) \in \mathbf{Z}_+ \times \mathbf{Z} : n + x = \text{even}\}$, where $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. If we let $\eta_n(x) = 1$ if $x \in \eta_n$ and $= 0$ if $x \notin \eta_n$, then the above evolution can be rewritten as

$$\eta_{n+1}^A(x) = \begin{cases} \eta_n^A(x+1) + \eta_n^A(x-1) \quad \bmod 2 & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases}$$

We call this process the *cancellative model* in this paper, since it has 'cancellative duality'. See pp 114–23 in Durrett (1988) for details.

When $f(2) = q$ with $0 \leqslant q \leqslant 1$, this more general class was first studied by Domany and Kinzel (1984), so it is often called the Domany–Kinzel model. Concerning this class, the reader is referred to Durrett (1988), pp 90–8, for example. In this setting, the directed bond percolation ($q = 2p - p^2$) and the directed site percolation ($q = p$) are special cases. The mixed site–bond directed percolation with the probability of open site α and with the probability of open bond β corresponds to the case of $p = \alpha\beta$ and $q = \alpha(2\beta - \beta^2)$.

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When $0 \leqslant p \leqslant q \leqslant 1$, the process is called attractive and has the following nice property: if $\eta_n^A \subset \eta_n^B$, then we can guarantee that $\eta_{n+1}^A \subset \eta_{n+1}^B$ for any $n \geqslant 0$ by using an appropriate coupling. However, the cancellative model (i.e. $q = 0$) is non-attractive, so it does not have the above property.

We let η_n^0 be the cancellative model at time n starting from the origin. Here we introduce a survival probability for it:

$$\theta(p) = P(\eta_n^0 \neq \emptyset \text{ for any } n \geqslant 0).$$

The sequence of events $\{\eta_n^0 \neq \emptyset\}$ is decreasing, so $\theta(p)$ is well defined. We introduce two critical probabilities as follows:

$$p_c = \sup\{p \in [0,1] : \theta(p') = 0 \text{ for any } p' \in [0, p]\}$$

$$p_c^* = \inf\{p \in [0,1] : \theta(p') > 0 \text{ for any } p' \in [p, 1]\}.$$

The above definitions give

$$0 \leqslant p_c \leqslant p_c^* \leqslant 1$$

since $\theta(0) = 0$ and $\theta(1) = 1$. Note that it is not proved whether or not $\theta(p)$ is a non-decreasing function in p, since the cancellative model under consideration is not attractive. However, Monte Carlo simulations suggest that the above monotonicity is valid; that is, it is conjectured that $p_c = p_c^*$. The estimated value is $p_c \approx 0.82$ by Kinzel (1985) using finite-size scaling calculations.

The present paper is devoted to the best rigorous lower bounds on the critical probabilities p_c and p_c^*.

Here we review some known results on lower bounds for p_c and p_c^*.

It is easy to see that $0.5 \leqslant p_c$ by comparison with a branching process Z_n as follows. Each particle gives rise to Y particles in the next generation where Y is given by

$$P(Y = 2) = p^2 \qquad P(Y = 1) = 2p(1-p) \qquad P(Y = 0) = (1-p)^2.$$

If $|\eta_n| = k$, then $E|\eta_{n+1}| \leqslant E|Z_{n+1}| = 2kp$. So, if $p < 0.5$, then $E(|\eta_{n+1}||\eta_n)/|\eta_n| \leqslant 2p < 1$ and the model will eventually die out. Note that this argument does not depend on attractiveness.

To obtain an improved lower bound on p_c^*, define the survival probability from the finite set $A \subset 2\mathbf{Z}$ as

$$\sigma(A) = P(\eta_n^A \neq \emptyset \text{ for any } n \geqslant 0).$$

By using the Harris lemma (see Harris 1976), Konno (1997) gave the following upper bound on $\sigma(A)$ for finite A. Let

$$p_c^{(K)} = \inf\{p \in [0,1] : 2p^3 - 2p^2 + 2p - 1 \geqslant 0\} = 0.647\,799\ldots.$$

For any $p \in [p_c^{(K)}, 1]$, we have

$$\sigma(A) \leqslant 1 - \alpha_*^{|A|}\beta_*^{b(A)} \qquad \text{for all} \quad A \in Y \tag{1.1}$$

where $|A|$ is the cardinality of A, $b(A)$ is the number of neighbouring pairs in A,

$$\alpha_* = \frac{p^4 - 2p^3 + 2p^2 - 2p + 1}{p^4} \qquad \text{and} \qquad \beta_* = \left(\frac{p\alpha_* + 1 - p}{\alpha_*}\right)^2.$$

In particular, if we take $A = \{0\}$, then we have

$$\theta(p) \leqslant \frac{2p^3 - 2p^2 + 2p - 1}{p^4} \qquad (p \in [p_c^{(K)}, 1]) \tag{1.2}$$

$$p_c^{(K)} = 0.647\,799\ldots \leqslant p_c^*. \tag{1.3}$$

Furthermore, assuming a relation (see equation (1.8)), we can obtain an improved lower bound on p_c (not p_c^*) by making a comparison between an annihilating branching process (a cancellative dual process for the cancellative model) and a coalescing branching process (a coalescing dual process for directed site percolation). More detailed discussions can be found on pp 119–20 of Durrett (1988).

Here we give the outline of this story. We let $N_n^x(y)$ be the number of paths from $(x, 0)$ to (y, n). Define ξ_n^A by the directed site percolation at time n starting from A. So we have

$$\xi_n^A(x) = \min\{N_n^A(x), 1\} \qquad \eta_n^A(x) = N_n^A(x) \mod 2.$$

The cancellative dual process $\tilde{\eta}_n$ for cancellative model η_n is an annihilating branching process. A particle at site x at time n branches into two particles put at $x+1$ and $x-1$ at time $n+1$ with probability p, and dies with no children with probability $1-p$. If two particles give birth onto the same site, their two offspring annihilate each other and an empty site results. On the other hand, the coalescing dual process $\tilde{\xi}_n$ for directed site percolation ξ_n is a coalescing branching process. A particle at site x at time n branches into two particles put at $x+1$ and $x-1$ at time $n+1$ with probability p, and dies with no children with probability $1-p$. If two particles give birth onto the same site, their two offspring coalesce into one. So the above observation implies for any $n \geqslant 0$

$$\tilde{\eta}_n^A \subset \tilde{\xi}_n^A \tag{1.4}$$

by using an appropriate coupling. By cancellative and coalescing duality equations, respectively, we have

$$P(|\eta_n^A \cap B| \text{ is odd}) = P(|\tilde{\eta}_n^B \cap A| \text{ is odd}) \tag{1.5}$$

and

$$P(\xi_n^A \cap B \neq \emptyset) = P(\tilde{\xi}_n^B \cap A \neq \emptyset). \tag{1.6}$$

Let ξ_n^1 denote directed site percolation starting from $\xi_0^1 = 2\mathbf{Z}$ and $\eta_n^{1/2}$ denote the cancellative system starting from a product measure with density $\frac{1}{2}$. From (1.4)–(1.6),

$$P(0 \in \xi_n^1) = P(\tilde{\xi}_n^0 \neq \emptyset) \geqslant P(\tilde{\eta}_n^0 \neq \emptyset) = 2P(0 \in \eta_n^{1/2}). \tag{1.7}$$

In the case of directed site percolation, we introduce two critical values:

$$p_e = \sup\{p : \xi_\infty^1 = \delta_\emptyset\} \qquad p_f = \sup\{p : P(\xi_n^0 \neq \emptyset \text{ for any } n \geqslant 0) = 0\}$$

where $\xi_\infty^1 = \lim_{n\to\infty} \xi_n^1$ and $\delta_\emptyset$ is the pointmass on $\emptyset$. The attractiveness gives $p_e = p_f$. It is known that the estimated value of p_e is 0.706 (see p 120 of Durrett 1988, for example).

If the following is valid:

$$p_c = \sup\{p : \eta_\infty^{1/2} = \delta_\emptyset \text{ for any } p' \in [0, p]\} \tag{1.8}$$

then we conclude that $p_c \geqslant p_e$ (≈ 0.706) by (1.7). However, the validity of (1.8) is not proved.

On the other hand, concerning the upper bound on p_c^*, Bramson and Neuhauser (1994) proved that

$$p_c^* < 1$$

by using a rescaling argument. Their basic idea is to show that the model for p close enough to 1, when viewed on a suitable length and time scale dominates a supercritical directed site percolation. So the existence of the phase transition is established rigorously.

Here we summarize the last parts of known results as we mentioned before: under the condition (1.8),

$$p_e(\approx 0.706) \leqslant p_c \leqslant p_c^* < 1.$$

In this situation, we give the following improved lower bounds on p_c and p_c^* by the method for finding suitable supermartingales for the model.

Theorem 1.1. *The cancellative model dies out when* $p \leqslant 0.771$. *It also dies out for values of* $p = 0.777$ *and* 0.781. *So we have* $0.771 \leqslant p_c$ *and* $0.781 \leqslant p_c^*$.

In the next section we will introduce the supermartingale method and give a proof of theorem 1.1 by using it.

2. The supermartingale method and proof of theorem 1.1

This method has been described in Sudbury (1998, 1999) where it was applied to processes in continuous time. It needs very little adaptation here. We shall begin with the simplest case. Assume that the occupied set is finite. We shall look for values of p for which the process tends to contract, and thus die out.

Let the rightmost particle of η be in position r. Then we define a score for the process:

$$S(\eta) = r + S_i$$

where i denotes the state of the configuration to the left of r and the S_i are a set of values to be determined. In this, the simplest case, $i = 0$ if $\eta(r-2) = 0$ and $i = 1$ if $\eta(r-2) = 1$. (Later when we consider the n positions to the left of r, i will range over 2^n possible values.) Given p we aim to find a choice of S_i such that $S(\eta)$ is a supermartingale in the sense that $E(S(\eta_{n+1})) \leqslant S(\eta_n)$.

Without loss of generality, we take $S_0 = 0$, $S_1 = -s$. To determine the change in expectation we may sometimes need to know the whole configuration of η. When we do not, we assume the situation most favourable to an increase in the expectation. We consider the possible changes at the right-hand end for four possibilities. Designate $E(S(\eta_{n+1})) - S(\eta_n)$ by $\delta(\eta)$.

Case 1. Right-hand end is... 0.0.1.0...

$$\delta(\eta) \leqslant p[p(1-s) + (1-p)(1)] + (1-p)[p(-1)] + (1-p)^2(-5). \qquad (2.1)$$

The first term is when both {10} pairs produce a 1 between them, making r increase by 1 and the configuration jump to state 1 (a 1 to the left of the rightmost 1). Then we consider the other three possibilities for 1's or 0's between the two {01} pairs. Note that -5 is the smallest possible reduction in r when two 0's appear.

Case 2. Right-hand end is... 1.0.1.0...

$$\begin{aligned}\delta(\eta) \leqslant\ & p[p(1-s) + (1-p)(1)] + (1-p)p[p(-1-s) + (1-p)(-1)] \\ & +(1-p)^2[p(-3) + (1-p)(-5)]. \qquad (2.2)\end{aligned}$$

It is simple to check that this exceeds the previous $\delta(\eta)$ by $p(1-p)(2-p(2+s))$. We shall assume that this is positive now and check it later. $\delta(\eta) \leqslant 0$ for both cases 1 and 2 is thus just the inequality for case 2, which is equivalent to

$$s \geqslant \frac{-5 + 12p - 8p^2 + 2p^3}{2p^2 - p^3}. \tag{2.3}$$

Case 3. Right-hand end is... 1.1.1.0...

$$\delta(\eta) \leqslant p(1+s) + (1-p)(-5+s). \tag{2.4}$$

Note that we have assumed the best outcome if the right-hand {10} produces a 0, that is, that r only decreases by 5 and that the process jumps to state $i = 0$.

Case 4. Right-hand end is... 0.1.1.0...

$$\delta(\eta) \leqslant p(1+s) + (1-p)[p(-3+s) + (1-p)(-5+s)]. \tag{2.5}$$

The right-hand side for case 4 exceeds that for case 3 by $p(1-p)$. $\delta(\eta) \leqslant 0$ for both cases 3 and 4 is thus just the inequality for case 4 which is equivalent to

$$s \leqslant 5 - 8p + 2p^2. \tag{2.6}$$

It is simple to check that $(-5 + 12p - 8p^2 + 2p^3)/(2p^2 - p^3)$ is increasing on $(0, 1]$ and that $5 - 8p + 2p^2$ is decreasing, and that both inequalities for s are satisfied for $p \in (0, 0.711)$ where $s = 0.32$. Having checked that $2 - p(2+s) > 0$, the assumption we made earlier, we have shown:

Lemma 2.1. *The cancellative model dies out for $p \leqslant 0.711$.*

Given a value of p we may also put an upper bound on the edge-speed $\delta(\eta)$. From equations (2.2) and (2.5) we may derive the following:

Lemma 2.2. *Given a value of p, the edge-speed for a cancellative model is bounded above by either side of the equation*

$$-5 + 12p - 8p^2 + 2p^3 - (2p^2 - p^3)s = -5 + 8p - 2p^2 + s$$

where s is chosen to make the two sides equal.

Table 1 shows upper bounds for the edge-speed for various values of p.

Table 1.

p	Edge-speed
0.7	−0.05
0.6	−0.47
0.5	−0.95

The smaller p, the worse these bounds become as the restriction on only retreating five spaces becomes a worse and worse approximation.

It is possible to improve on the bounds above by considering more positions to the left of the rightmost particle than just the one we have above. If we consider n positions (in intervals of 2) then there are 2^n such configurations, numbered $i = 0, \ldots, 2^n - 1$. We shall associate with these configurations a set of values $\{S_i\}$ which are to be determined. The probability of going from state i to state j, to be designated q_{ij}, may well depend on the configuration to the left of the n positions being considered, but we shall always assume the configuration to the left most favourable for rightwards spread of the process. In state i we designate a_i to be the expected change in the position of the rightmost particle. With these most favourable configurations, the expected change in the score

$$E(S(\eta_{n+1})) - S(\eta_n) \leqslant q_{ij}(S_i - S_j) + a_i. \tag{2.7}$$

It can be seen that the above analysis of the case $n = 1$ is in this form.

Given p, the task is therefore to find a set S_i which will make the right-hand side of (2.7) $\leqslant 0$ for all pairs i, j in which case the process must die out. The procedure for doing this is explained in Sudbury (1998), even though the IPS considered there evolved in continuous time.

Table 2 shows the maximum values of p for which η dies out for various values of n.

Table 2.

n	p
2	0.727
3	0.752
4	0.764
5	0.773
6	0.778
7	0.782

Up to this point our analysis has allowed us to see that for certain values of p it is possible to find a set $\{S_i^p\}$ such that $\sum_j q_{ij}(S_j^p - S_i^p) + a_i$ is negative for all i for all possible Q-matrices $\{q_{ij}\}$, and thus that the process must die out. What we now need to do is to show that the values of p in between have the same property.

First let us fix the number of sites. For each particular number of sites n there was a maximum value of p, to be called p_n, for which we could show that the process died out. We now use the set $\{S_i^{p_n}\}$ for all the values of p in (p_{n-1}, p_n). (These values of $\{S_i\}$ are usually not optimal for all the p in this interval but may be good enough. Luckily, they usually are.)

We shall proceed from p_{n-1} to p_n by jumps which are sufficiently small to show that all values of p between the jumps will have $\sum_j q_{ij}(S_j^{p_n} - S_i^{p_n}) + a_i$ negative for all possible $\{q_{ij}\}$. (Remember that the range of possibilities for the $\{q_{ij}\}$ is determined by the set of possible configurations to the left of the n sites.) The maximum value of $\sum_j q_{ij}(S_j^{p_n} - S_i^{p_n}) + a_i$ is to be called summax(p), the p dependency residing in q and a. Suppose summax(p_1), summax$(p_2) < 0$ with $p_1 < p_2$. summax(p) could only be positive in (p_1, p_2) if summax(p) rose to 0 and then decreased to summax(p_2). At some point in (p_1, p_2) the derivative of summax(p) would need to be $<$ summax$(p_2)/(p_2 - p_1)$. It is clear that a_i increases with p and therefore cannot contribute to a negative derivative. $|(S_j^{p_n} - S_i^{p_n})| < \max(S_k^{p_n}) - \min(S_k^{p_n})$, a bound which is fixed since the $S_k^{p_n}$ do not change through these tests.

Suppose that in state i there are r neighbouring pairs of type 01. These are the pairs of sites which determine the rates q_{ij}. We shall show that $|\mathrm{d}/\mathrm{d}p(q_{ij})| < 1.3\sqrt{r}$ and thus that:

Theorem 2.3. *If $r_{\max}$ is the maximum number of pairs of sites involved in the calculations of q_{ij}, then*

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[\sum_j q_{ij}(S_j - S_i) + a_i\right] > -1.3\sqrt{r_{\max}}(\max(S_k^{p_n}) - \min(S_k^{p_n})).$$

We prove this theorem using the following lemmas.

Lemma 2.4.

$$\frac{\mathrm{d}}{\mathrm{d}p}\sum_{x=l}^{r}\binom{r}{x}p^{r-x}(1-p)^{x} = -\frac{r!}{(l-1)!(r-l)!}p^{r-l}(1-p)^{l-1} = rP(X=r-l)$$

where $X \sim \text{binomial}(r-1, p)$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}p}\binom{r}{x}p^{r-x}(1-p)^{x} = \frac{r!}{x!(r-x-1)!}p^{r-x-1}(1-p)^{x} - \frac{r!}{(x-1)!(r-x)!}p^{r-x}(1-p)^{x-1}.$$

Only one term remains after cancellation. □

Now it is well known that the maximum binomial probability for fixed r, p occurs as close to rp as possible and is asymptotically $1/\sqrt{2\pi rpq}$. The error comes from Stirling's approximation and is less than $\exp[(r^{-1} + (rp)^{-1} + (rq)^{-1})/12] < \mathrm{e}^{1/4} < 1.3$. We have $0.6 < p < 0.8$, giving $1/\sqrt{2\pi pq} < 0.997$, so that

$$\frac{r!}{(l-1)!(r-l)!}p^{r-l}(1-p)^{l-1} = rP(X=r-l) < 1.3\sqrt{r}.$$

Lemma 2.5. *The derivative of a binomial series with parameters* r, $0.6 < p < 0.8$ *with some terms censored has an absolute value* $< 1.3\sqrt{r}$, *or*

$$-1.3\sqrt{r} < \frac{\mathrm{d}}{\mathrm{d}p}\sum_{x=0}^{r} I(x)\binom{r}{x}p^{r-x}(1-p)^{x} < 1.3\sqrt{r}$$

where $I(x) = 0$ *or* 1.

Proof. The derivatives of the individual terms in the binomial series start all positive and then become all negative. It is clear that the largest negative value of the derivative of the censored series occurs when all negative and no positive terms are included; that is the series given in lemma 2.1 where l is chosen to be the first term with a negative derivative. We have shown above that the size of this expression is $< 1.3\sqrt{r}$. The same argument for positive terms applies as for negative terms. □

We wish to bridge the gaps between the values for which we have determined $\text{summax}(p)$. Theorem 2.3 shows that we need to choose the gap between trials, $p_2 - p_1 < |\,\text{summax}(p_2)|/(1.3\sqrt{r_{\text{max}}}(\max(S_k) - \min(S_k)))$. Using four sites we may then proceed from $p = 0.711$ to 0.75 with gaps of 0.002, then from 0.75 to 0.762 with gaps varying from 0.001 to 0.0002. Using five sites we may then proceed from $p = 0.762$ to 0.771 with gaps varying from 0.0002 to 0.000 02. With six sites our luck runs out and the S_i obtained from $p_6 = 0.777$ do not deliver negative values of $\text{summax}(p)$.

So it is obtained that the process dies out when $p \leqslant 0.771$. It also dies out for the values of $p = 0.777$ and 0.781. (It has not proved possible to test for all values between 0.771 and 0.781.) Those complete the proof of theorem 1.1.

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